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# New nonrenormalization theorems for anomalous three point functions

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Abstract: Nonrenormalization theorems involving the transverse, i.e. non anomalous, part of the $\langle VVA \rangle$ correlator in perturbative QCD are proven. Some of the consequences and questions they raise are discussed.

Keywords: Nonperturbative Effects, Spontaneous Symmetry Breaking, Anomalies in Field and String Theories, QCD.



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## Contents



## 1. Introduction

Since their discovery more than thirty years ago [1, 2], anomalous Ward identities for three point functions involving vector and axial currents have been studied quite extensively and from various points of view [3] (see also refs. [4] and [5]). In QCD with three massless flavours, these anomalous contributions appear in the Green's functions involving the conserved Noether currents of the global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ chiral symmetry. Most remarkable in this context is the property that the expression of these anomalous contributions is protected from perturbative QCD corrections, so that it takes the same form as in a theory of free quarks [6, 7, 8, 9]. Strictly speaking, no similar statement is available at the nonperturbative level. However, the argument of ref. [10] requires that, in an appropriate normalization, the coefficient of the anomaly be an integer, $N_C$ in the case of QCD. This makes it very likely that the anomaly is also preserved after non perturbative corrections. Assuming that QCD confines, and that confinement does indeed not modify the coefficient of the anomaly, then entails that the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ chiral symmetry of QCD is necessarily spontaneously broken towards its diagonal $\mathrm{SU}(3)_V$ subgroup [11, 12, 13].

These properties are well established and concern the longitudinal part of the corresponding Green's functions, which is thus completely fixed by the Adler-Bell-Jackiw anomaly. The transverse parts however are not affected, and satisfy the naive Ward identities. Therefore, the *new* result that perturbative QCD corrections to the free quark contribution are also absent in the *transverse* part of the $\langle VVA\rangle$ three-point function came as rather unexpected [14, 15]. Although this was obtained only in a very specific kinematical limit, as we shall see later on, it is also true even in a more general setting.

The interest in this QCD correlator stems from the fact that it appears in the determination of the hadronic contribution to a class of two loop weak corrections to the anomalous magnetic moment of the muon. As has been first emphasized in ref. [16], the contribution from the light $u$, $d$ and $s$ quarks to the transverse part of the $\langle VVA\rangle$ triangle involves properties of QCD of a nonperturbative nature. A detailed evaluation, within the framework of



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large-$N_c$ QCD, has been published in ref. [17]. However, a recent reanalysis by the authors of ref. [15] finds also a perturbative contribution to this transverse part, which originates in a discrepancy between the treatment of the Operator Product Expansion done in [17] and the one in [15]. Although this discrepancy results in a numerical effect which is too small to influence the present comparison between theory and experiment in the muon $g-2$, the underlying theoretical issues involved in this discrepancy are of theoretical interest and, therefore, deserve special attention.

Let us consider the theorem on the non-renormalization of the transverse parts of the $\langle VVA\rangle$ Green's function discovered in [14]. Actually, what we shall find is that there is a whole class of theorems, of which the one considered in ref. [14] represents a special case. We shall also provide an explanation as to the origin and interpretation of these results. Before proceeding, let us recall a few useful facts concerning anomalous Ward Identities (see also the discussion in ref. [18]).

The Ward identities describing the invariance properties of massless QCD under the transformations of the global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ chiral symmetry group are most conveniently obtained upon considering the transformation properties of the generating functional $Z[v,a]$, defined as

$$\begin{aligned} e^{iZ[v,a]} &= \int \mathcal{D}[G]\mathcal{D}[\psi,\overline{\psi}]e^{i\int d^4x \mathcal{L}_{\mathrm{QCD}}(v,a)} \\ &= \int \mathcal{D}[G]e^{i\int d^4x\mathcal{L}_{YM}}\det D\,, \end{aligned} \tag{1.1}$$

with $D=\gamma^\mu(\partial_\mu - iG_\mu - iv_\mu - i\gamma_5 a_\mu)$ the Dirac operator in presence of the external vector and axial sources for the SU(3) flavour currents,[1] and in the presence of the gluon field configuration $G_\mu$. The determinant of the Dirac operator that appears, after integration over the quark fields, in the second expression for $Z[v,a]$ needs to be regularized. The $\zeta$-function technique[2] offers a convenient regularization,

$$\ln\det{}_\mu D = -\frac{1}{2}\frac{d}{ds}\frac{\mu^{2s}}{\Gamma(s)}\int_0^\infty d\lambda\,\lambda^{s-1}\,\mathrm{Tr}\,e^{-\lambda\overline{D}^2}\bigg|_{s=0}, \tag{1.2}$$

with $\overline{D} = \gamma_5 D$. Other regularizations of the functional determinant, det $D$, may be considered. They differ from the previous one by a local functional of the sources of dimension less than or equal to four,

$$\ln\det{}_F D = \ln\det{}_\mu D - \int d^4x\, F(v,a)\,. \tag{1.3}$$

Equivalently, these changes of the regularization may be considered as modifications, by local counterterms, of the chronological products of the $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ currents. Two

[1] We restrict ourselves to nonsinglet flavour currents, i.e. the sources $v_\mu$ and $a_\mu$ are singlets in colour space and traceless matrices in flavour space.

[2] In order to apply this type of regularization, one first has to perform a rotation to euclidean space. We assume that this has been done, but we give the resulting expressions after having rotated back to Minkowski space.



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particular definitions of the fermion determinant are of interest. The first one, $\ln \det_{AB} D$, reproduces the Adler-Bardeen form of the anomaly [21] and corresponds to the choice

$$F_{AB}(v,a) = -i\,\frac{N_C}{12\pi^2}\,\mathrm{tr}_f\left\{\frac{1}{2}\,\nabla_\mu a_\nu \nabla^\mu a^\nu - 3iv^{\alpha\beta}a_\alpha a_\beta - a_\alpha a_\beta a^\alpha a^\beta + 2a_\alpha a^\alpha a_\beta a^\beta\right\}. \quad (1.4)$$

The second definition reproduces a factorized or left-right symmetric form of the anomaly [21], $\ln \det_{\mathrm{LR}} D$, and corresponds to the choice

$$F_{\mathrm{LR}}(v,a) = F_{AB}(v,a) - \frac{N_C}{24\pi^2}\,\epsilon^{\alpha\beta\mu\nu}\;\mathrm{tr}_f\left\{iv_{\mu\nu}\{v_\alpha, a_\beta\} - a_\alpha a_\beta a_\mu v_\nu - v_\alpha v_\beta v_\mu a_\nu\right\}. \quad (1.5)$$

Naively, one would expect $Z[v,a]$ to be invariant, $Z[v,a] = Z[v+\delta v, a+\delta a]$, under the following variations of the sources

$$\begin{aligned}\delta v_\mu &= \partial_\mu \alpha + i[\alpha, v_\mu] + i[\beta, a_\mu]\\ \delta a_\mu &= \partial_\mu \beta + i[\alpha, a_\mu] + i[\beta, v_\mu]\,,\end{aligned} \quad (1.6)$$

where the Lie algebra valued functions $\alpha(x) = \alpha^a(x)\lambda^a/2$ and $\beta(x) = \beta^a(x)\lambda^a/2$ correspond to infinitesimal vector and axial transformations, respectively. Indeed, this specific variation of the sources can in principle be compensated by a change of variables in the quark fields corresponding to a local $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ gauge transformation. However, the functional measure of integration $\mathcal{D}[\psi, \overline{\psi}]$ over the quark fields is not invariant [19, 20] under such a change of variables. Instead, there appears a non trivial jacobian, which leads to

$$\delta \ln \det_{AB} D = -i\,\frac{N_C}{(4\pi)^2}\int d^4x\, \mathrm{tr}_f(\beta\Omega)\,, \quad (1.7)$$

with

$$\Omega = \epsilon^{\alpha\beta\mu\nu}\left[v_{\alpha\beta}v_{\mu\nu} + \frac{4}{3}\nabla_\alpha a_\beta \nabla_\mu a_\nu + \frac{2i}{3}\{v_{\alpha\beta}, a_\mu a_\nu\} + \frac{8i}{3}a_\mu v_{\alpha\beta} a_\nu + \frac{4}{3}a_\alpha a_\beta a_\mu a_\nu\right], \quad (1.8)$$

and

$$\delta \ln \det_{\mathrm{LR}} D = -i\,\frac{N_C}{(4\pi)^2}\int d^4x\, \mathrm{tr}_f[(\alpha+\beta)A(F^R) + (\alpha-\beta)A(F^L)]\,, \quad (1.9)$$

where $F^R_\mu = (v+a)_\mu$, $F^L_\mu = (v-a)_\mu$, and

$$A(F) = \frac{1}{3}\,\epsilon^{\alpha\beta\mu\nu}\left[2\partial_\alpha F_\beta \partial_\mu F_\nu - i\partial_\alpha(F_\beta F_\mu F_\nu)\right]. \quad (1.10)$$

## 2. Non renormalization theorems for $\langle VVA \rangle$

Let us now consider, for $a,b,c = 3,8$, the QCD three point functions

$$\begin{aligned}\mathcal{W}^{abc}_{\mu\nu\rho}(q_1,q_2) &= i\int d^4x_1 d^4x_2\, e^{i(q_1\cdot x_1 + q_2\cdot x_2)} \times \langle\, 0\,|\,\mathrm{T}\{V^a_\mu(x_1)V^b_\nu(x_2)A^c_\rho(0)\}\,|\,0\,\rangle\\ &\equiv \frac{1}{2}\,d^{abc}\,\mathcal{W}_{\mu\nu\rho}(q_1,q_2)\,,\end{aligned} \quad (2.1)$$

of the colour singlet light flavour currents

$$V_\mu^a = \overline{\psi}\gamma_\mu \frac{\lambda^a}{2}\psi\,, \qquad A_\mu^a = \overline{\psi}\gamma_\mu\gamma_5 \frac{\lambda^a}{2}\psi\,, \qquad \psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}. \tag{2.2}$$

Taking $q_1$ or $q_2$ to be the small momentum going through an external electromagnetic field, this Green's function appears in the hadronic electroweak contributions to the muon $g-2$ at the two-loop level [17, 15]. To be more precise, in the muon $g-2$ there is also the contribution from the flavour singlet component of the Z boson, but we have not included this flavour singlet in the definition of the Green's function (2.1) in an attempt to simplify the discussion which will follow and because it does not play any crucial role.

Besides $\mathcal{W}_{\mu\nu\rho}$ in eq. (2.1), we shall also need

$$\begin{aligned}\Omega_{\mu\nu\rho}^{abc}(q_1,q_2) &= i\int d^4x_1 d^4x_2\, e^{i(q_1\cdot x_1+q_2\cdot x_2)} \times \langle\, 0\,|\, \mathrm{T}\{L_\mu^a(x_1)V_\nu^b(x_2)R_\rho^c(0)\}\,|\, 0\,\rangle \\ &\equiv \frac{1}{2}\, d^{abc}\,\Omega_{\mu\nu\rho}(q_1,q_2)\,,\end{aligned} \tag{2.3}$$

with

$$L_\mu^a = \overline{\psi_L}\gamma_\mu \frac{\lambda^a}{2}\psi_L\,, \qquad R_\mu^a = \overline{\psi_R}\gamma_\mu \frac{\lambda^a}{2}\psi_R\,, \qquad \psi_{L,R} = \frac{1}{2}(1\mp\gamma_5)\psi\,. \tag{2.4}$$

The two Green's functions $\mathcal{W}_{\mu\nu\rho}$ and $\Omega_{\mu\nu\rho}$ are actually related. Use of charge-conjugation invariance allows one to do away with the combinations $\langle VVV\rangle$ and $\langle VAA\rangle$ in the function $\Omega_{\mu\nu\rho}$ and obtain that, in fact,

$$\Omega_{\mu\nu\rho}(q_1,q_2) = \frac{1}{4}\Big[\mathcal{W}_{\mu\nu\rho}(q_1,q_2) - \mathcal{W}_{\rho\nu\mu}(-q_1-q_2,q_2)\Big]. \tag{2.5}$$

In the above definitions, the time ordering corresponds to the definition of the chronological product T which preserves the conservation of the vector currents, i.e. to the prescription $\det_{AB} D$. If instead we wish to use the prescription $\det_{\mathrm{LR}} D$, the corresponding chronological product will be denoted by $\widehat{\mathrm{T}}$. For instance,

$$\begin{aligned}\widehat{\Omega}_{\mu\nu\rho}^{abc}(q_1,q_2) &= i\int d^4x_1 d^4x_2\, e^{i(q_1\cdot x_1+q_2\cdot x_2)} \times \langle\, 0\,|\, \widehat{\mathrm{T}}\{L_\mu^a(x_1)V_\nu^b(x_2)R_\rho^c(0)\}\,|\, 0\,\rangle \\ &\equiv \frac{1}{2}\, d^{abc}\,\widehat{\Omega}_{\mu\nu\rho}(q_1,q_2)\,,\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\widehat{\mathcal{W}}_{\mu\nu\rho}^{abc}(q_1,q_2) &= i\int d^4x_1 d^4x_2\, e^{i(q_1\cdot x_1+q_2\cdot x_2)} \times \langle\, 0\,|\, \widehat{\mathrm{T}}\{V_\mu^a(x_1)V_\nu^b(x_2)A_\rho^c(0)\}\,|\, 0\,\rangle \\ &\equiv \frac{1}{2}\, d^{abc}\,\widehat{\mathcal{W}}_{\mu\nu\rho}(q_1,q_2)\,.\end{aligned} \tag{2.7}$$

The relation between $\widehat{\Omega}_{\mu\nu\rho}$ and $\widehat{\mathcal{W}}_{\mu\nu\rho}$ is the same as the one between $\Omega_{\mu\nu\rho}$ and $\mathcal{W}_{\mu\nu\rho}$.

Furthermore,

$$\begin{aligned}\mathcal{W}_{\mu\nu\rho}^{abc}(q_1,q_2) &= \widehat{\mathcal{W}}_{\mu\nu\rho}^{abc}(q_1,q_2) + \\ &\quad + i\int d^4x_1 d^4x_2 d^4z\, e^{i(q_1\cdot x_1+q_2\cdot x_2)}\frac{1}{i^3}\,\frac{\delta^3[F_{\mathrm{LR}}(v,a)-F_{AB}(v,a)](z)}{\delta v^{a\mu}(x_1)\delta v^{b\nu}(x_2)\delta a^{c\rho}(0)}\bigg|_{v=a=0}\,,\end{aligned}$$

which gives

$$\mathcal{W}_{\mu\nu\rho}(q_1,q_2) = \widehat{\mathcal{W}}_{\mu\nu\rho}(q_1,q_2) + \frac{N_C}{12\pi^2}\,\epsilon_{\mu\nu\rho\alpha}(q_1-q_2)^\alpha\,, \tag{2.8}$$

whereas

$$\Omega_{\mu\nu\rho}(q_1,q_2) = \widehat{\Omega}_{\mu\nu\rho}(q_1,q_2) - \frac{N_C}{16\pi^2}\epsilon_{\mu\nu\rho\tau}q_2^\tau\,. \tag{2.9}$$

The Ward identities satisfied by these correlators read (we work in the chiral limit),

$$\begin{aligned}
\{q_1^\mu\,;\,q_2^\nu\}\,\mathcal{W}_{\mu\nu\rho}(q_1,q_2) &= \{0\,;\,0\}\\
(q_1+q_2)^\rho\,\mathcal{W}_{\mu\nu\rho}(q_1,q_2) &= -\frac{N_C}{4\pi^2}\,\epsilon_{\mu\nu\sigma\tau}q_1^\sigma q_2^\tau\,,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
q_1^\mu\,\Omega_{\mu\nu\rho}(q_1,q_2) &= -\frac{N_C}{16\pi^2}\,\epsilon_{\mu\nu\rho\tau}q_1^\mu q_2^\tau\\
q_2^\nu\,\Omega_{\mu\nu\rho}(q_1,q_2) &= 0\\
(q_1+q_2)^\rho\,\Omega_{\mu\nu\rho}(q_1,q_2) &= -\frac{N_C}{16\pi^2}\,\epsilon_{\mu\nu\sigma\tau}q_1^\sigma q_2^\tau\,,
\end{aligned} \tag{2.11}$$

whereas

$$\{q_1^\mu\,;\,q_2^\nu\,;\,(q_1+q_2)^\rho\}\,\widehat{\Omega}_{\mu\nu\rho}(q_1,q_2) = \{0\,;\,0\,;\,0\}\,. \tag{2.12}$$

Let us now discuss the transformation properties of these various quantities under the action of the $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ flavour group. The ordinary product $L^a(x)V^b(y)R^c(0)$ transforms under the representation $(\mathbf{8}_L\otimes\mathbf{8}_L,\mathbf{8}_R)\oplus(\mathbf{8}_L\,,\,\mathbf{8}_R\otimes\mathbf{8}_R)$, which does not project onto the singlet representation $(\mathbf{1}_L,\mathbf{1}_R)$. The same property is still true for the canonical chronological product of these currents, defined with the help of the step function in the time variables. However, this does not lead to a covariant time ordering. On the other hand, a covariant chronological product is likely to introduce $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ invariant contributions. This is precisely not the case for the chronological product $\widehat{\mathrm{T}}$, which rests on the prescription $\det_{\mathrm{LR}}$ for the fermionic determinant. Indeed, with this choice of time ordering, the three point function $\widehat{\Omega}_{\mu\nu\rho}(q_1,q_2)$ satisfies the naive Ward identities. Anomalous contributions then occur only in the Ward identities for the three or four point functions involving only left currents or only right currents. Therefore, $\widehat{\mathrm{T}}\{L^a(x)V^b(y)R^c(0)\}$ has the same transformation properties as the ordinary product, and thus $\widehat{\Omega}_{\mu\nu\rho}(q_1,q_2)$ is an order parameter of the $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ chiral symmetry, which means that it does not receive perturbative QCD corrections at any order, i.e.

$$\widehat{\Omega}_{\mu\nu\rho}(q_1,q_2)|_{\mathrm{pQCD}} = 0\,. \tag{2.13}$$

The Ward identities restrict the general decomposition of $\mathcal{W}_{\mu\nu\rho}(q_1,q_2)$ into invariant functions to four terms

$$\begin{aligned}
\mathcal{W}_{\mu\nu\rho}(q_1,q_2) = -\frac{1}{8\pi^2}\Big\{ &-w_L\left(q_1^2,q_2^2,(q_1+q_2)^2\right)(q_1+q_2)_\rho\,\epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta +\\
&+ w_T^{(+)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right)t^{(+)}_{\mu\nu\rho}(q_1,q_2) +\\
&+ w_T^{(-)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right)t^{(-)}_{\mu\nu\rho}(q_1,q_2) +\\
&+ \widetilde{w}_T^{(-)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right)\widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2)\Big\}\,,
\end{aligned} \tag{2.14}$$

with the transverse tensors

$$
\begin{aligned}
t^{(+)}_{\mu\nu\rho}(q_1,q_2) &= q_{1\nu}\epsilon_{\mu\rho\alpha\beta}q_1^\alpha q_2^\beta - q_{2\mu}\epsilon_{\nu\rho\alpha\beta}q_1^\alpha q_2^\beta - (q_1\cdot q_2)\epsilon_{\mu\nu\rho\alpha}(q_1-q_2)^\alpha + \\
&\quad + \frac{q_1^2+q_2^2-(q_1+q_2)^2}{(q_1+q_2)^2}\,\epsilon_{\mu\nu\alpha\beta}\;q_1^\alpha q_2^\beta (q_1+q_2)_\rho\,, \\
t^{(-)}_{\mu\nu\rho}(q_1,q_2) &= \left[(q_1-q_2)_\rho - \frac{q_1^2-q_2^2}{(q_1+q_2)^2}\,(q_1+q_2)_\rho\right]\epsilon_{\mu\nu\alpha\beta}\;q_1^\alpha q_2^\beta \\
\widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2) &= q_{1\nu}\epsilon_{\mu\rho\alpha\beta}q_1^\alpha q_2^\beta + q_{2\mu}\epsilon_{\nu\rho\alpha\beta}\;q_1^\alpha q_2^\beta - (q_1\cdot q_2)\epsilon_{\mu\nu\rho\alpha}\;(q_1+q_2)^\alpha\,.
\end{aligned}
\tag{2.15}
$$

Bose symmetry entails

$$
\begin{aligned}
w_T^{(+)}\left(q_2^2,q_1^2,(q_1+q_2)^2\right) &= +w_T^{(+)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right) \\
w_T^{(-)}\left(q_2^2,q_1^2,(q_1+q_2)^2\right) &= -w_T^{(-)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right) \\
\widetilde{w}_T^{(-)}\left(q_2^2,q_1^2,(q_1+q_2)^2\right) &= -\widetilde{w}_T^{(-)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right)\,.
\end{aligned}
\tag{2.16}
$$

In addition, the longitudinal part is entirely fixed by the anomaly,

$$
w_L\left(q_1^2,q_2^2,(q_1+q_2)^2\right) = -\frac{2N_C}{(q_1+q_2)^2}\,. \tag{2.17}
$$

A straightforward computation then leads to

$$
\begin{aligned}
\widehat{\Omega}_{\mu\nu\rho}(q_1,q_2) = -\frac{1}{32\pi^2}\Bigg\{ & -\epsilon_{\mu\nu\rho\alpha}\;q_2^\alpha\left[q_1^2\;w_L\left((q_1+q_2)^2,q_2^2,q_1^2\right)+2N_C\right] + \\
& + \Big[q_1^2 w_L\left((q_1+q_2)^2,q_2^2,q_1^2\right) - \\
& \quad - (q_1+q_2)^2 w_L\left(q_1^2,q_2^2,(q_1+q_2)^2\right)\Big]\frac{(q_1+q_2)_\rho}{(q_1+q_2)^2}\epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta + \\
& + t^{(+)}_{\mu\nu\rho}(q_1,q_2)\Bigg[w_T^{(+)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right) - \frac{1}{2}w_L\left((q_1+q_2)^2,q_2^2,q_1^2\right) + \\
& \qquad + \frac{(q_2^2+q_1\cdot q_2)}{q_1^2}\;w_T^{(+)}\left((q_1+q_2)^2,q_2^2,q_1^2\right) - \\
& \qquad - \left(\frac{q_1\cdot q_2}{q_1^2}+1\right)w_T^{(-)}\left((q_1+q_2)^2,q_2^2,q_1^2\right)\Bigg] + \\
& + t^{(-)}_{\mu\nu\rho}(q_1,q_2)\Bigg[w_T^{(-)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right) + \frac{1}{2}w_L\left((q_1+q_2)^2,q_2^2,q_1^2\right) - \\
& \qquad - \frac{(q_1^2+q_2^2+q_1\cdot q_2)}{q_1^2}\;w_T^{(+)}\left((q_1+q_2)^2,q_2^2,q_1^2\right) + \\
& \qquad + \frac{q_1\cdot q_2}{q_1^2}w_T^{(-)}\left((q_1+q_2)^2,q_2^2,q_1^2\right)\Bigg] + \\
& + \widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2)\Bigg[\widetilde{w}_T^{(-)}\left(q_1^2,q_2^2,(q_1+q_2)^2\right) + \widetilde{w}_T^{(-)}\left((q_1+q_2)^2,q_2^2,q_1^2\right) - \\
& \qquad - \frac{1}{2}\,w_L\left((q_1+q_2)^2,q_2^2,q_1^2\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(q_1^2+q_2^2+q_1\cdot q_2)}{q_1^2}w_T^{(+)}\left((q_1+q_2)^2,q_2^2,q_1^2\right)-\\
&-\left(\frac{q_1\cdot q_2}{q_1^2}-1\right)w_T^{(-)}\left((q_1+q_2)^2,q_2^2,q_1^2\right)\Bigg]\Bigg\}. \qquad (2.18)
\end{aligned}
$$

Due to the expression (2.17) of $w_L\left(q_1^2,q_2^2,(q_1+q_2)^2\right)$, the two first terms on the right-hand side of eq. (2.18) vanish identically. Since the three tensors (2.15) are independent, the property (2.13) implies that the combinations of invariant functions that multiply $t^{(+)}_{\mu\nu\rho}(q_1,q_2)$, $t^{(-)}_{\mu\nu\rho}(q_1,q_2)$ and $\widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2)$ in the expression (2.18) have to vanish to all orders in perturbation theory. Consequently, the following three non renormalization theorems follow,

$$
\left\{\left[w_T^{(+)}+w_T^{(-)}\right]\left(q_1^2,q_2^2,(q_1+q_2)^2\right)-\left[w_T^{(+)}+w_T^{(-)}\right]\left((q_1+q_2)^2,q_2^2,q_1^2\right)\right\}_{\mathrm{pQCD}}=0 \qquad (2.19)
$$

$$
\left\{\left[\widetilde{w}_T^{(-)}+w_T^{(-)}\right]\left(q_1^2,q_2^2,(q_1+q_2)^2\right)+\left[\widetilde{w}_T^{(-)}+w_T^{(-)}\right]\left((q_1+q_2)^2,q_2^2,q_1^2\right)\right\}_{\mathrm{pQCD}}=0 \qquad (2.20)
$$

and

$$
\begin{aligned}
&\left\{\left[w_T^{(+)}+\widetilde{w}_T^{(-)}\right]\left(q_1^2,q_2^2,(q_1+q_2)^2\right)+\left[w_T^{(+)}+\widetilde{w}_T^{(-)}\right]\left((q_1+q_2)^2,q_2^2,q_1^2\right)\right\}_{\mathrm{pQCD}}-\\
&\qquad -w_L\left((q_1+q_2)^2,q_2^2,q_1^2\right)= \qquad (2.21)\\
&=-\left\{\frac{2(q_2^2+q_1\cdot q_2)}{q_1^2}w_T^{(+)}\left((q_1+q_2)^2,q_2^2,q_1^2\right)-2\,\frac{q_1\cdot q_2}{q_1^2}w_T^{(-)}\left((q_1+q_2)^2,q_2^2,q_1^2\right)\right\}_{\mathrm{pQCD}},
\end{aligned}
$$

involving the transverse part of the $\langle VVA\rangle$ correlator $\mathcal{W}_{\mu\nu\rho}(q_1,q_2)$, and which hold for all values of the momentum transfers $q_1^2$, $q_2^2$ and $(q_1+q_2)^2$. Notice that in eq. (2.21) the longitudinal function $w_L$ does not need to carry the subindex "pQCD" due to the nonrenormalization of the anomaly.

The non renormalization theorem obtained in refs. [14, 15] appears as a particular case. Indeed, upon taking $q_1=k\pm q$, $q_2=-k$, and keeping only the terms linear in the momentum $k$, one readily obtains

$$
\begin{aligned}
t^{(+)}_{\mu\nu\rho}(k\pm q,-k)&=q^2\epsilon_{\mu\nu\rho\sigma}k^\sigma-q_\mu\epsilon_{\nu\rho\alpha\beta}q^\alpha k^\beta-q_\rho\epsilon_{\mu\nu\alpha\beta}q^\alpha k^\beta+\mathcal{O}(k^2)\\
t^{(-)}_{\mu\nu\rho}(k\pm q,-k)&=\mathcal{O}(k^2)\\
\widetilde{t}^{(-)}_{\mu\nu\rho}(k\pm q,-k)&=q^2\epsilon_{\mu\nu\rho\sigma}k^\sigma-q_\mu\epsilon_{\nu\rho\alpha\beta}q^\alpha k^\beta-q_\rho\epsilon_{\mu\nu\alpha\beta}q^\alpha k^\beta+\mathcal{O}(k^2)\,. \qquad (2.22)
\end{aligned}
$$

Within this same kinematical configuration, the three non renormalization theorems then reduce to one single equality, namely the result of refs. [14, 15]

$$
w_L(Q^2)=2w_T(Q^2)_{\mathrm{pQCD}}\,, \qquad (2.23)
$$

where $Q^2\equiv -q^2$ and

$$
\begin{aligned}
w_L(Q^2)&=w_L(-Q^2,0,-Q^2)\\
w_T(Q^2)&=w_T^{(+)}(-Q^2,0,-Q^2)+\widetilde{w}_T^{(-)}(-Q^2,0,-Q^2)\,. \qquad (2.24)
\end{aligned}
$$



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Our result (2.13) thus contains and extends the non renormalization theorem of refs. [14, 15] to general values of the momentum transfers. More interestingly perhaps, it identifies the origin and the meaning of these results: they merely follow from the fact that, with an appropriate definition of the chronological product, the $\langle LLR\rangle$ three point correlator (2.3) is an order parameter of the chiral symmetry group of QCD with three massless flavours. The proof of the particular case (2.23) outlined in ref. [14] relies on arguments of a diagrammatic kind. In appendix A, we show that a proof of (2.13) can also be established along these lines. The proof is based on the same argumentation as that of refs. [6, 9] in the case of the chiral anomaly (see also appendix B).

## 3. The function $w_L(Q^2) - 2w_T(Q^2)$ in full QCD

Using the general decomposition of $\widehat{\Omega}_{\mu\nu\rho}(q_1, q_2)$ in eq. (2.18), we can relate the amplitude $w_L(Q^2) - 2w_T(Q^2)$ to the three point function $\widehat{\Omega}_{\mu\nu\rho}(q_1, q_2)$ in the appropriate kinematic configuration where $q_1 = k \pm q$, $q_2 = -k$, and only the terms linear in $k$ are kept, namely

$$\begin{aligned}\widehat{\Omega}_{\mu\nu\rho}(k\pm q, -k) = \ & \frac{1}{32\pi^2}\left[w_L(Q^2) - 2w_T(Q^2)\right] \times \\ & \times \left(q^2\epsilon_{\mu\nu\rho\sigma}k^\sigma - q_\mu\epsilon_{\nu\rho\alpha\beta}q^\alpha k^\beta - q_\rho\epsilon_{\mu\nu\alpha\beta}q^\alpha k^\beta\right) + \mathcal{O}(k^2)\,. \end{aligned} \tag{3.1}$$

The result of eq. (2.23) is nothing but the statement that $\widehat{\Omega}_{\mu\nu\rho}$ vanishes in perturbation theory.[3] Contracting this expression with $\epsilon^{\mu\nu\rho\lambda}$, we have

$$\lim_{k\to 0}\frac{\partial}{\partial k^\lambda}\left(\epsilon^{\mu\nu\rho\lambda}\widehat{\Omega}_{\mu\nu\rho}(k\pm q, -k)\right) = \frac{3}{8\pi^2}\, Q^2\left[w_L(Q^2) - 2w_T(Q^2)\right], \tag{3.2}$$

or equivalently, using the definition of $\widehat{\Omega}_{\mu\nu\rho}(q_1, q_2)$ in eq. (2.6),

$$Q^2\left[w_L(Q^2) - 2w_T(Q^2)\right] = \frac{16\pi^2}{\sqrt{3}}\int d^4x\int d^4y\, e^{iq\cdot x}(y-x)_\lambda\epsilon^{\mu\nu\rho\lambda}\langle 0|\widehat{\mathrm{T}}\{L^3_\mu(x)V^3_\nu(y)R^8_\rho(0)\}|0\rangle\,, \tag{3.3}$$

explicitly showing the fact that the function $Q^2\left[w_L(Q^2) - 2w_T(Q^2)\right]$ is an order parameter of S$\chi$SB at all values of its argument. Obviously, in the chiral limit, perturbation theory produces a vanishing result for the right-hand side of eq. (3.3) to all orders. Furthermore, since the function $w_L(Q^2)$ is exact [21], it follows that $w_T(Q^2)$ cannot receive contributions in pQCD beyond its lowest order value. However, this all-orders result for $w_T(Q^2)$ in perturbation theory cannot go through at the nonperturbative level. Indeed, the low-$Q^2$ behaviour of $w_T(Q^2)$ is governed by the $\mathcal{O}(p^6)$ effective chiral lagrangian in the odd-parity sector. The relevant coupling is the term (see ref. [22] for notations)

$$\mathcal{L}^W_6 = C^W_{22}\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(u^\mu\{\nabla_\gamma f^{\gamma\nu}_+, f^{\alpha\beta}_+\}\right) + \cdots\,, \tag{3.4}$$

which fixes $w_T(0)$ in terms of the low-energy constant $C^W_{22}$ as follows

$$w_T(0) = 128\pi^2 C^W_{22}\,. \tag{3.5}$$

[3] The reader who has found our derivation of eq. (3.1) a bit too formal is referred to appendix B for a less formal version of it.

Unfortunately, there is no model independent information on this constant. Therefore, contrary to the case of the anomalous amplitude $w_L(Q^2)$ which is fixed by the $\mathcal{O}(p^4)$ Wess-Zumino term in the effective chiral lagrangian, the transverse amplitude $w_T(Q^2)$ *has no pole* at $Q^2 = 0$. This clearly shows that $w_T(Q^2)$ is affected by *nonperturbative* QCD corrections, in contrast to the case of $w_L(Q^2)$.[4] At large values of $Q^2$ one can start seeing this different behaviour through the use of the Operator Product Expansion. In this way, both the authors of ref. [17] and [15, 14] agree that $w_T(Q^2)$ receives, as the leading nonperturbative contribution at large $Q^2$, a term proportional to

$$\left\{w_T(Q^2)\right\}_{NP} \sim \frac{\alpha_s \langle\overline{\psi}\psi\rangle^2}{Q^6 F_\pi^2}\,. \tag{3.6}$$

Since the function $w_L(Q^2)$ is exactly given by its perturbative result, i.e. $2N_C/Q^2$, eqs. (3.5) and (3.6) break the perturbative degeneracy $w_L(Q^2) = 2\ w_T(Q^2)$. Notice as well that $w_T(Q^2)$ can only receive nonperturbative contributions in the Operator Product Expansion coming from operators which are order parameters of spontaneous chiral symmetry breaking. For instance, a contribution from the gluon condensate $\langle G^2\rangle/Q^6$ is excluded. This operator was wrongly allowed in the analysis of ref. [15], although it was then numerically neglected on the basis of being accompanied by a one-loop suppressed Wilson coefficient.

The main difference between the analysis of ref. [17] and [15, 14] lies in the existence (or not) of a $1/Q^2$ contribution to the function $w_T(Q^2)$ at large values of $Q^2$. The analysis of ref. [15, 14] was based on a separation of virtuality in momentum and found a contribution which goes like $w_T(Q^2) = N_C/Q^2$ from the region of high virtuality in the *perturbative* quark loop. In ref. [17], on the other hand, no contribution of $\mathcal{O}(1/Q^2)$ was found. The calculation in [17] was based on the fact that $w_T(Q^2)$ satisfies an unsubtracted dispersion relation and that the high-energy contribution to $\mathrm{Im}\, w_T$ from the perturbative QCD continuum vanishes in the chiral limit.[5] Closely related to this point is the interpretation of the anomaly as a subtraction in the corresponding dispersion relation for $w_L(Q^2)$ [3].

As we have discussed after eq. (3.3), there is no doubt that $w_T(Q^2) = N_C/Q^2$ in perturbation theory. The question is whether this is also true in the exact theory, for large enough values of $Q^2$. The result of ref. [17] rests on the assumption that the leading large-$Q^2$ contribution to the function $w_T(Q^2)$ comes from the region of large momentum in its perturbative imaginary part (i.e. from the $q\overline{q}$ continuum). This assumption is based on the notion of (global) duality between quarks and hadrons [23]; notion which is heavily based on examples such as the process $e^+e^- \to$ hadrons.

On the other hand, the analysis of ref. [15, 14] is based on a separation of virtuality in momentum in the perturbative quark loop, assuming that this carries through also at the nonperturbative level. Even though a proof of the Operator Product Expansion is still lacking in QCD, the $1/Q^2$ contribution obtained in [15, 14] is difficult to avoid since it originates from the region of infinitely short distances. This is completely unlike the case

[4]The argument which requires $w_L(Q^2)$ to be normalized by an integer [10] also does not apply to $w_T(Q^2)$.
[5]See the appendix D in ref. [17].



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of $e^+e^- \to$ hadrons because this $1/Q^2$ behavior, although being perturbative, cannot be associated with any $q\overline{q}$ continuum in the chiral limit and this explains why the $1/Q^2$ was not found in [17]. In the light of this discussion, we are ready to accept the result of [15, 14] at this point, even though we feel that a better understanding of the role played by the $q\overline{q}$ continuum would be desirable.

We find, however, that the properties of the function $w_T(Q^2)$ are then rather intriguing and in the rest of this note we would like to point out several of the "curiosities" which, we think, may deserve further investigation. Firstly, upon taking the limit $q \to 0$ in (3.3), one obtains

$$N_C = \frac{8\pi^2}{\sqrt{3}} \int d^4x \int d^4y\, (y-x)_\lambda \epsilon^{\mu\nu\rho\lambda} \langle 0|\widehat{\mathrm{T}}\{L^3_\mu(x) V^3_\nu(y) R^8_\rho(0)\}|0\rangle\,, \tag{3.7}$$

since $w_L(Q^2)$ has a pole at $Q^2 = 0$ (see eq. (2.17)) but $w_T(Q^2)$ is regular. We find this result quite amazing as it equates the residue of the anomaly — a term which can be computed in perturbation theory — to a Green's function which measures the spontaneous breakdown of chiral symmetry. It would be very interesting to check this result by nonperturbative methods such as, for instance, lattice gauge theories.

Related to the previous discussion is the fact that all the hadronic QCD sum rules we know which originate in pQCD short-distance properties are of the type

$$\int_0^\infty dt\, \omega(t, M^2)\rho(t) \sim N_C \left[1 + \mathcal{O}\left(\frac{\alpha_s(M^2)}{\pi}\right)\right], \tag{3.8}$$

where $\rho(t)$ denotes some generic spectral function and $\omega(t, M^2)$ an appropriate weight function, e.g., $\omega(t, M^2) = \exp(-t/M^2)$ in the so called Laplace or Borel QCD sum rules. In all these sum rules, it is the presence of $\alpha_s(M^2)$ corrections which controls the regime in the euclidean at which we can trust the pQCD calculations. By contrast, the behaviour

$$\lim_{Q^2\to\infty} w_T(Q^2) = \frac{N_C}{Q^2}\,, \qquad \text{with no } \alpha_s(Q^2) \text{ corrections}\,, \tag{3.9}$$

leaves us with no scale to gauge the validity of the asymptotic pQCD behaviour.

In fact, eq. (3.9) implies an exact QCD sum rule of a new type:

$$\int_0^\infty dt\, \frac{1}{\pi} \mathrm{Im}\, w_T(t) = N_C\,, \tag{3.10}$$

which follows from the fact that $w_T(q^2)$ obeys an unsubtracted dispersion relation. The sum rule (3.10) thus clearly shows a marked difference with respect to, say, the first Weinberg sum rule,

$$\int_0^\infty dt\, \frac{1}{\pi} \left[\mathrm{Im}\, \Pi_V(t) - \mathrm{Im}\, \Pi_A(t)\right] = F_\pi^2\,, \tag{3.11}$$

where *both sides* receive subleading corrections in the $1/N_C$ expansion. The striking feature about the sum rule in eq. (3.10) is that the r.h.s. is exact. There are no $1/N_C$ corrections on the r.h.s., while the l.h.s., which is an integral of hadronic contributions, has certainly subleading terms in the $1/N_C$ expansion; e.g., those generated by multiparticle states. For these subleading terms, this implies a very curious fine tuning of the various terms on the l.h.s. which have to add up to zero.

There is also and exact QCD sum rule for the $w_L(Q^2)$ amplitude

$$\int_0^\infty dt\, \frac{1}{\pi}\, \mathrm{Im}\, w_L(t) = 2\ N_C\,, \tag{3.12}$$

which, in the hadronic spectrum, is fulfilled by the Goldstone pole alone: $\frac{1}{\pi}\,\mathrm{Im}\, w_L(t) = 2N_C\delta(t)$. Therefore, there is nothing surprising about this sum rule, which just reflects the fact that the anomaly is an exact result. By contrast, the sum rule in eq. (3.10) implies a very subtle fine tuning between an infinite number of couplings and masses of the hadronic spectrum and the anomaly. As we have seen, there is no clear cut argument which would allow one to simply dismiss the presence of the free quark $N_C/Q^2$ short distance contribution in $w_T(Q^2)$. However, from our experience gained from the study of other QCD Green's functions, its persistence beyond the perturbative regime leads to rather peculiar properties of $w_T(Q^2)$.

We hope to have presented enough evidence that the combination $w_L(Q^2) - 2\ w_T(Q^2)$, i.e. eq. (3.3), is a very interesting object for the study of nonperturbative issues in QCD such as the spontaneous breakdown of chiral symmetry, the Operator Product Expansion and the chiral anomaly. We hope that some of the above observations will help attract further attention on the understanding of its properties.

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## A. Perturbative proof of the nonrenormalization theorems

The perturbative expansion of $\mathcal{W}_{\mu\nu\rho}(q_1, q_2)$ is given by

$$\mathcal{W}_{\mu\nu\rho}(q_1, q_2)|_{\mathrm{pQCD}} = (-1) i^4\, \frac{N_C}{2} \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \mathcal{W}^{[n]}_{\mu\nu\rho}(q_1, q_2)\,. \tag{A.1}$$

The lowest order contribution arises from the free quark triangle, and reads

$$\mathcal{W}^{[0]}_{\mu\nu\rho}(q_1, q_2) = \Gamma^{[0]}_{\mu\nu\rho}(q_1, q_2|a) + \Gamma^{[0]}_{\nu\mu\rho}(q_2, q_1|b)\,, \tag{A.2}$$

with

$$\Gamma^{[0]}_{\mu\nu\rho}(q_1, q_2|a) = \int \frac{d^4P}{(2\pi)^4}\, \mathrm{tr}\left[\frac{1}{\not{P} - \not{q}_1 + \not{a} - m_q}\, \gamma_\mu\, \frac{1}{\not{P} + \not{a} - m_q}\, \gamma_\nu\, \frac{1}{\not{P} + \not{q}_2 + \not{a} - m_q}\, \gamma_\rho \gamma_5\right]. \tag{A.3}$$

In these expressions, the arbitrary four-vectors $a_\mu$ and $b_\mu$ parametrize the ambiguity in defining the routing of the loop momentum $P_\mu$. This ambiguity reads

$$\Gamma^{[0]}_{\mu\nu\rho}(q_1, q_2|a) = \Gamma^{[0]}_{\mu\nu\rho}(q_1, q_2) - \frac{1}{8\pi^2}\, \epsilon_{\mu\nu\rho\sigma} a^\sigma\,, \tag{A.4}$$

where $\Gamma^{[0]}_{\mu\nu\rho}(q_1,q_2)$ corresponds to $\Gamma^{[0]}_{\mu\nu\rho}(q_1,q_2|a)$ with the choice $a_\mu = (0,0,0,0)$. Therefore,

$$\mathcal{W}^{[0]}_{\mu\nu\rho}(q_1,q_2) = \Gamma^{[0]}_{\mu\nu\rho}(q_1,q_2) + \Gamma^{[0]}_{\nu\mu\rho}(q_2,q_1) - \frac{1}{8\pi^2}\,\epsilon_{\mu\nu\rho\sigma}(a-b)^\sigma \tag{A.5}$$

The difference $(a-b)_\mu$ is fixed upon imposing the conservation of the vector current. For $m_q = 0$, one has

$$\begin{aligned}
(q_1+q_2)^\rho \Gamma^{[0]}_{\mu\nu\rho}(q_1,q_2) &= 0 \\
q_1^\mu \left[\Gamma^{[0]}_{\mu\nu\rho}(q_1,q_2) + \Gamma^{[0]}_{\nu\mu\rho}(q_2,q_1)\right] &= -\frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma} q_1^\mu q_2^\sigma \\
q_2^\nu \left[\Gamma^{[0]}_{\mu\nu\rho}(q_1,q_2) + \Gamma^{[0]}_{\nu\mu\rho}(q_2,q_1)\right] &= -\frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma} q_1^\nu q_2^\sigma \,.
\end{aligned} \tag{A.6}$$

Therefore, $\{q_1^\mu ; q_2^\nu\}\mathcal{W}^{[0]}_{\mu\nu\rho}(q_1,q_2) = \{0;0\}$ provided $(a-b)_\mu = 2(q_1-q_2)_\mu$, which, for massless quarks, then leads to

$$(q_1+q_2)^\rho \mathcal{W}^{[0]}_{\mu\nu\rho}(q_1,q_2) = \frac{1}{2\pi^2}\,\epsilon_{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma \,. \tag{A.7}$$

Staying in the chiral limit, one has, upon shifting the loop momentum

$$\begin{aligned}
\Gamma^{[0]}_{\mu\nu\rho}(q_1,q_2) &= \Gamma^{[0]}_{\nu\rho\mu}(q_2,-q_1-q_2|q_2) \\
&= \Gamma^{[0]}_{\nu\rho\mu}(q_2,-q_1-q_2) - \frac{1}{8\pi^2}\,\epsilon_{\mu\nu\rho\sigma} q_2^\sigma \,.
\end{aligned} \tag{A.8}$$

Consequently,

$$\mathcal{W}^{[0]}_{\mu\nu\rho}(q_1,q_2) - \mathcal{W}^{[0]}_{\rho\nu\mu}(-q_1-q_2,q_2) = \frac{1}{2\pi^2}\,\epsilon_{\mu\nu\rho\sigma} q_2^\sigma \,. \tag{A.9}$$

As far as the higher order contributions $\mathcal{W}^{[n]}_{\mu\nu\rho}(q_1,q_2)$, $n \geq 1$, are concerned, they may be written as

$$\mathcal{W}^{[n]}_{\mu\nu\rho}(q_1,q_2) = \Gamma^{[n]}_{\mu\nu\rho}(q_1,q_2) + \Gamma^{[n]}_{\nu\mu\rho}(q_2,q_1) \,. \tag{A.10}$$

The integration over the triangle loop is now well defined, provided all the subgraphs arising from the QCD corrections (for instance quark and gluon self energies) have been properly regularized, i.e. assuming one works in the same conditions which allow to prove the Adler-Bardeen non renormalization theorem for the anomalous part. Then all shifts in the momenta are allowed, and one has

$$\Gamma^{[n]}_{\mu\nu\rho}(q_1,q_2) = \Gamma^{[n]}_{\nu\rho\mu}(q_2,-q_1-q_2) \,. \tag{A.11}$$

Therefore,

$$\mathcal{W}^{[n]}_{\mu\nu\rho}(q_1,q_2) - \mathcal{W}^{[n]}_{\rho\nu\mu}(-q_1-q_2,q_2) = 0 \,. \tag{A.12}$$

Taking (A.9), (A.12), (2.5) and (2.9) leads to

$$\widehat{\Omega}_{\mu\nu\rho}|_{\mathrm{pQCD}} = 0 \,. \tag{A.13}$$

This by itself does not prove, but is certainly compatible with the fact that $\widehat{\Omega}_{\mu\nu\rho}$ is an order parameter. However, it suffices to entail the non renormalization theorems discussed before.



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## B. The function $\Omega_{\mu\nu\rho}$ in the one-family standard model

Let us consider the first family of quarks *and* leptons in the Standard Model. In this case, the currents appearing in the Green's function $\Omega_{\mu\nu\rho}$ in eq. (2.3) are the complete ones, i.e. including not only the $u, d$ quarks but also the electron and the neutrino. These currents are made up with the generators of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ subgroup of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ which gets gauged when the electroweak interactions are turned on. Indeed, this is the Green's function which contributes to physical observables such as the muon $g-2$.[6] In the case where $q_1 = k \pm q$, $q_2 = -k$ and $k$ is small, keeping only terms linear in this momentum $k$ and working to lowest order in the electroweak interactions, one finds that

$$\Omega_{\mu\nu\rho}(k \pm q, -k) = \frac{1}{32\pi^2}\Big\{ w_L^{\mathrm{TOTAL}}(Q^2)\left(-q_\rho \epsilon_{\mu\nu\alpha\sigma} q^\alpha k^\sigma - q_\mu \epsilon_{\nu\rho\alpha\sigma} q^\alpha k^\sigma\right) + \\ + 2w_T^{\mathrm{TOTAL}}(Q^2)\left(-q^2 \epsilon_{\mu\nu\rho\sigma} k^\sigma + q_\mu \epsilon_{\nu\rho\alpha\sigma} q^\alpha k^\sigma + q_\rho \epsilon_{\mu\nu\alpha\sigma} q^\alpha k^\sigma\right)\Big\}. \tag{B.1}$$

However, in this case the functions $w_{L,T}^{\mathrm{TOTAL}}(Q^2)$ contain quarks as well as leptons. Therefore,

$$w_L^{\mathrm{TOTAL}}(Q^2) = w_L^{\mathrm{quarks}}(Q^2) + w_L^{\mathrm{leptons}}(Q^2) = 0\,, \tag{B.2}$$

as a consequence of the anomaly cancellation in the Standard Model. For the function $w_T^{\mathrm{TOTAL}}$ one obtains

$$2w_T^{\mathrm{TOTAL}}(Q^2) = 2w_T^{\mathrm{quarks}}(Q^2) + 2w_T^{\mathrm{leptons}}(Q^2)\,, \tag{B.3}$$

but, obviously, $w_L^{\mathrm{leptons}}(Q^2) = 2w_T^{\mathrm{leptons}}(Q^2)$ to all orders in $\alpha_s$ because leptons do not experience strong interactions. Consequently,

$$2w_T^{\mathrm{TOTAL}}(Q^2) = 2w_T^{\mathrm{quarks}}(Q^2) + w_L^{\mathrm{leptons}}(Q^2)\,, \tag{B.4}$$

and using eq. (B.2), one finally obtains

$$2w_T^{\mathrm{TOTAL}}(Q^2) = 2w_T^{\mathrm{quarks}}(Q^2) - w_L^{\mathrm{quarks}}(Q^2)\,, \tag{B.5}$$

which, together with eq. (B.1), is the result in eq. (3.1) in the text.

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